

**SECOND ORDER CORRECTIONS OF THE
POISSON BOOTSTRAP**

G. Jogesh Babu, P.K. Pathak, and C.R. Rao

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SECOND ORDER CORRECTIONS OF THE POISSON BOOTSTRAP

G. JOGESH BABU, P.K. PATHAK, AND C.R. RAO

ABSTRACT. Rao, Pathak and Koltchinskii (1997) have recently studied a sequential approach to resampling in which resampling is carried out sequentially one-by-one (with replacement each time) until the bootstrap sample contains $m \approx (1 - e^{-1})n \approx .632n$ distinct observations from the original sample. In our previous work, we have established that the main empirical characteristics of the sequential bootstrap go through, in the sense of being within a distance of order $O(n^{-3/4})$ from those of the usual bootstrap. However, the theoretical justification of the second order correctness of the sequential bootstrap is somewhat difficult. It is the main topic of this investigation. Among other things, we accomplish it by approximating our sequential scheme by a resampling scheme based on the Poisson distribution with mean $\mu = 1$ and censored at $X = 0$.

1. INTRODUCTION

Efron (1979) introduced the bootstrap method of resampling as a ubiquitous sampling technique of estimating the variance of an estimator and sampling distribution of a given statistic. In a fundamental paper, Bhattacharya and Ghosh (1978) have demonstrated that Edgeworth expansion for a wide class of statistics can be derived from Edgeworth expansions for multivariate sample means. This technique has been used by Singh (1981) to show, in the case of univariate sample mean, that the bootstrap is more accurate than the central limit theorem when higher order population moments exist. These ideas are further exploited by Babu and Singh (1983, 1984) to show the superiority of the bootstrap method, and by Babu and Singh (1985) to obtain Edgeworth expansions for the ratio statistic and similar statistics based on samples from finite populations. The method is also used by Babu and Singh (1989)

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to obtain global Edgeworth expansions for functions of means of random vectors, when one of the coordinates has a lattice distribution and the remaining part of the vector has a strongly non-lattice distribution. Later, Gene and Zinn (1990) showed that in a certain weak sense, the bootstrap method is valid (consistent) if and only if the central limit theorem holds. In fact the central limit theorem furnishes accuracy of approximation of order $o(1)$, while if the third population moment exists, one can expect, in many commonly encountered populations the accuracy of the bootstrap method to be of order $o(n^{-1/2})$, where n denotes the sample size. Thus while the bootstrap method has the potential of being second-order accurate, the central limit approximation is not so. This is one of the several reasons for the current interest and preference in the literature for those methods of resampling that are second-order accurate, i.e., accurate of the order $o(n^{-1/2})$.

Stemming from Efron's observation that the information content of a bootstrap sample is based on approximately $(1 - e^{-1})100\% \approx 63\%$ of the original sample, Rao, Pathak and Koltchinskii (1997) have introduced a sequential resampling method in which sampling is carried out one-by-one (with replacement) until $(m + 1)$ distinct original observations appear, where m denotes the largest integer not exceeding $(1 - e^{-1})n$. It has been shown that the empirical characteristics of this sequential bootstrap are within a distance of order $O(n^{-3/4})$ from the usual bootstrap. The authors provide a heuristic argument in favor of their sampling scheme and establish the consistency of the sequential bootstrap; however the question of second-order correctness was not addressed.

The main object of this paper is to examine the second order correctness of the sequential bootstrap. The theoretical justification of this is somewhat more difficult because of the dependence among the bootstrap sample units. At this time a rigorous Edgeworth expansion under this kind of dependence is unavailable in the literature. A cumbersome approach based on computation of cumulants, under the (unsubstantiated) assumption that a formal Edgeworth expansion is valid, may be given along the lines of Hall and Mammen (1994). This does not lead to a complete solution as the

Edgeworth expansions are not known. Instead we first approximate the sequential bootstrap by another sequential resampling scheme based on the Poisson distribution. Under the new scheme the “independence” of sample units under resampling is preserved. A rigorous justification of the Edgeworth expansion can now be given more easily. This then provides a sound theoretical framework in which the second order correctness for the sequential bootstrap can be established. In this paper we provide details for the sample mean. Edgeworth expansions for statistics that can be represented as a smooth function of multivariate sample mean are considered in Section 4.

2. SEQUENTIAL RESAMPLING SCHEME

Let $S = (X_1, X_2, \dots, X_n)$ be a random sample from a distribution F , and $\theta(F)$ a parameter of interest. Let F_n denote the empirical distribution function based on S , and suppose that $\theta(F_n)$ is to be used as an estimator of $\theta(F)$. The Efron’s bootstrap method approximates the sampling distribution of a standardized version of $\sqrt{n}(\theta(F_n) - \theta(F))$ by the resampling distribution of a corresponding statistic $\sqrt{n}(\theta(\hat{F}_n) - \theta(F_n))$ based on a bootstrap sample \hat{S}_n in which the original F has been replaced by the empirical distribution based on the original sample S , and F_n of the former statistic has been replaced by the empirical distribution based on a bootstrap sample \hat{F}_n . In Efron’s bootstrap resampling scheme, $\hat{S}_n = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$ is a random sample of size n drawn from S by simple random sampling with replacement (SRSWR). In the Rao-Pathak-Koltchinskii (1997) sequential scheme, observations are drawn from S sequentially by SRSWR until there are $(m+1) = [n(1 - e^{-1})] + 2$ distinct original observations in the bootstrap sample; the last observation is discarded to ensure technical simplicity. Thus an observed bootstrap sample under the Rao-Pathak-Koltchinskii scheme admits the form:

$$\hat{S}_N = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_N) \tag{2.1}$$

in which $\hat{X}_1, \hat{X}_2, \dots, \hat{X}_N$ have $m \approx n(1 - e^{-1})$ distinct observations from S . The random sample size N admits the following decomposition in terms of the independent random variables:

$$N = I_1 + I_2 + \dots + I_m \quad (2.2)$$

in which $m = [n(1 - e^{-1})] + 1$; $I_1 = 1$, and for each k , $2 \leq k \leq m$,

$$P(I_k = j) = \left(1 - \frac{k-1}{n}\right) \left(\frac{k-1}{n}\right)^{j-1}. \quad (2.3)$$

Although we have established the consistency of this sampling scheme, a rigorous proof of its second order correctness requires an Edgeworth expansion for dependent random variables; such an expansion is unavailable in the literature at the present time. An alternative approach that can be used is to slightly modify the preceding resampling scheme so that existing techniques on Edgeworth expansion, such as those of Babu and Bai (1996), Bai and Rao (1991, 1992), Babu and Singh (1989) and others, can be employed. A modification of our previous resampling scheme that allows the second-order correctness to go through easily is as follows.

2.1. Poisson Bootstrap. For the selection of a bootstrap sample with a given number m of distinct units, under the *Poisson Bootstrap*, we provide a conceptual definition and a practical approach. Let us take a sample $\alpha_1, \dots, \alpha_n$ of n independent observations from $P(1)$, i.e., Poisson distribution with mean 1. If there are exactly m nonzero values in the sample, we accept it and take

$$\hat{S} = \{(X_1, \alpha_1), (X_2, \alpha_2), \dots, (X_n, \alpha_n)\}, \quad (2.4)$$

i.e., with the observation X_i repeated α_i times, as the bootstrap sample. If the number of nonzero values in $\alpha_1, \dots, \alpha_n$ is not exactly m , we reject the entire sample and draw another sample of size n . The bootstrap sample size N of \hat{S} as in (2.1) is a random variable

$$N = \alpha_1 + \dots + \alpha_n. \quad (2.5)$$

A practical way of implementing this resampling scheme is to first assign at random $(n - m)$ α 's a value of zero and to the remaining m α 's values independently chosen from the Poisson distribution with mean $\mu = 1$ and censored at $X = 0$. An outline of the equivalence of these two procedures is as follows.

Theorem 2.1. *The moment generating function $M_N(t)$ of N , the sample size of the Poisson resampling scheme, is given by*

$$M_N(t) = \left[\frac{(e^{(e^t-1)} - e^{-1})}{(1 - e^{-1})} \right]^m. \quad (2.6)$$

Proof. Let Y_1, Y_2, \dots, Y_n be n Poisson variables with mean $\mu = 1$. Then it is easily seen that

$$P(N = w) = \text{const} \left(\sum_1 \frac{e^{-n}}{\alpha_1! \alpha_2! \dots \alpha_m!} \right) \quad (2.7)$$

where the sum \sum_1 extends over all positive natural numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_m = w$. It then follows that (see Pathak (1961))

$$\begin{aligned} P(N = w) &= \text{const} \left[\frac{e^{-n}}{w!} (m^w - \binom{m}{1} (m! - 1)^w + \dots \pm 1^w) \right] \\ &= \text{const} \frac{e^{-n}}{w!} (\Delta^m X^w|_{X=0}) \end{aligned} \quad (2.8)$$

where Δ is the difference operator with unit increment.

From (2.8) it follows that

$$P(N = w) = \frac{1}{(e - 1)^m} \Delta^m \frac{X^w}{w!}|_{X=0}. \quad (2.9)$$

Consequently the moment generating function $M_N(t)$ of N is given by

$$\begin{aligned}
M_N(t) &= E(e^{tw}) = \sum_{w \geq 0} \frac{e^{tw}}{(e-1)^m} \Delta^m \frac{X^w}{w!} |_{X=0} \\
&= \frac{1}{(e-1)^m} \sum_{w \geq 0} \Delta^m \frac{(e^t X)^w}{w!} |_{X=0} \\
&= \Delta^m \frac{e^{X e^t}}{(e-1)^m} |_{X=0} \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(e-1)^m} \{ e^{m e^t} - \binom{m}{1} e^{(m-1)e^t} + \binom{m}{2} e^{(m-2)e^t} \dots \} \\
&= \frac{e^{m e^t}}{(e-1)^m} \{ 1 - \binom{m}{1} e^{-t} + \binom{m}{2} e^{-2t} - \dots \} \\
&= \frac{e^{m e^t}}{(e-1)^m} (1 - e^{-e^t})^m \\
&= \left[\frac{(e^{e^t} - 1)}{(e-1)} \right]^m \\
&= \left[\frac{(e^{(e^t-1)} - e^{-1})}{(1 - e^{-1})} \right]^m \tag{2.11}
\end{aligned}$$

This completes the proof. \square

The preceding theorem shows that the distribution of N can be viewed as that of the sum of m IID random variables with a common distribution with the moment generating function given by the formula

$$m(t) = \frac{(e^{(e^t-1)} - e^{-1})}{(1 - e^{-1})}. \tag{2.12}$$

It is evident that $m(t)$ is the moment generating function of the Poisson distribution with $\mu = 1$ and censored at $X = 0$. Let Y denote a random variable with moment generating function $m(t)$. Then $E(Y) = 1/(1 - e^{-1})$ and $V(Y) = e(e-2)/(e-1)^2$. Therefore

$$\begin{aligned}
E(N) &= mE(Y) \\
&= n + O(1) \tag{2.13}
\end{aligned}$$

and

$$\begin{aligned} V(N) &= mV(Y) \\ &= n(e - 2)/(e - 1) + O(1) \end{aligned} \quad (2.14)$$

2.2. Advantage of sequential bootstrap over regular bootstrap. One of the main advantages of the sequential bootstrap over the classical fixed sample size bootstrap is to avoid situations where a bootstrap sample has several repeated observations which may give rise to a degenerate value of the statistic under consideration.

Since the bootstrap utilizes all the data points, in general, the bootstrap estimator of variance of a statistic is not robust even for robust statistics. The bootstrap estimator of variance of the sample median m_n based on the sample X_1, \dots, X_n is given by

$$V_n^* = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (X_{(i)} - X_{(j)})^2 p_{i,n} p_{j,n},$$

where

$$X_{(1)} \leq \dots \leq X_{(n)}$$

is the ordering of the data,

$$p_{i,n} = n \binom{n-1}{r-1} \int_{(i-1)/n}^{i/n} u^r (1-u)^{n-r} du,$$

and $r = (n/2)$ if n is an even integer and $= (n+1)/2$ if it is an odd integer. Breakdown point is a widely used measure of robustness in modern statistical literature. The breakdown point of a statistic $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ is defined as (k/n) , where k is the minimum number of data points needed to be replaced by worst possible outliers to move the statistic beyond any bound. In the case of bootstrap variance of sample median, the breakdown point is $(1/n)$.

However, in the case of sequential bootstrap or Poisson bootstrap, $X_{(i)}$ for $i < m - (n/2)$ do not enter the estimate of the variance of the sample median. Hence the breakdown point is $\approx \frac{1}{2} - e^{-1} = 0.132$.

With these results, we now proceed to establish the second order correctness of the Poisson Bootstrap, a sequential bootstrap based on the Poisson distribution.

3. PRELIMINARY RESULTS

To establish Proposition 1 of Section 4, we require some preliminary results. First note that by Theorem 10.1 of Bhattacharya and Ranga Rao (1986), there exists a random variable V with distribution J such that $E(V^{10}) < \infty$, $P(|V| \geq 1) < \frac{1}{4}$, and the characteristic function \hat{J} of J vanishes outside the interval $[-c, c]$, for some $c > 0$, i.e.

$$\hat{J}(t) = 0 \quad \text{for } |t| > c \quad (3.1)$$

For any $\epsilon > 0$, let J_ϵ denote the distribution of $V\epsilon$. The next lemma is a trivial consequence of Lemma 24.1 of Bhattacharya and Ranga Rao (1986). The inequality is similar to (4.1) of Babu and Singh (1984). See also (4.2) of Babu and Singh (1984).

Lemma 1. *Suppose h is a Borel measurable function, on the real line \mathbb{R} , bounded by 1. Let μ be a finite measure and ν a finite signed measure. Then for any $0 < \epsilon < 1$,*

$$\begin{aligned} \left| \int h(y)(\mu - \nu)(dy) \right| &\leq 15 \left(|(\mu - \nu) * J_\epsilon|(\mathbb{R}) + (\mu(\mathbb{R}) + \nu^+(\mathbb{R}))P(|V| > \epsilon^{-1/2}) \right. \\ &\quad \left. + 3^{-\epsilon^{1/4}}(\mu(\mathbb{R}) + \nu^+(\mathbb{R})) \right. \\ &\quad \left. + \sup_{|z| \leq \epsilon^{1/4}} \int \omega(h, 2\epsilon, z + y)\nu^+(dy) \right), \end{aligned}$$

where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition, $|\nu| = \nu^+ + \nu^-$ is the total variation measure and

$$\omega(h, \epsilon, z) = \sup \left(|h(z + y') - h(z + y'')| : |y'| < \epsilon, |y''| < \epsilon \right).$$

Further we have for any $0 < v < 1$, $0 < \epsilon < 1$,

$$\begin{aligned} \int \omega(h, \epsilon, v - y) \phi(y) dy &\leq 3 \int \omega(h, \epsilon, y) \phi(y) dy \\ &\quad + O(v^{-1} \exp(-\frac{1}{8}v^{-2})). \end{aligned} \tag{3.2}$$

The inequality (3.2) is the relation (3.5) of Sweeney (1977).

The next lemma helps in estimating the total variation norm of $(\mu - \nu) * J_\epsilon$. Let $D^k h(t, s, \omega)$ denote the k -th partial derivative of h with respect to t .

Lemma 2. *Let g be a real valued function on $\mathbb{R} \times \mathbb{Z}^2$ satisfying*

$$\sum_{r=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int (1+x^2) |g(x, r, j)| dx < \infty.$$

Then for all integers r and j ,

$$\int_{-\infty}^{\infty} |g(x, r, j)| dx \leq \max_{k=0,2} \int_G \int_{-\infty}^{\infty} |D^k \hat{g}(t, s, \omega)| dt ds d\omega,$$

where $G = [-\pi, \pi] \times [-\pi, \pi]$ and \hat{g} denotes the Fourier transform of g , that is

$$\hat{g}(t, s, \omega) = \sum_{r=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} e^{-isr+i\omega j} \int_{-\infty}^{\infty} e^{itx} g(x, r, j) dx.$$

Proof. Following the proof of Lemma 11.6 of Bhattacharya and Ranga Rao (1986), let $A = \{x : g(x, r, j) \geq 0\}$. If for each r, j ,

$$\hat{g}(t; r, j) = \int_{-\infty}^{\infty} e^{itx} g(x, r, j) dx,$$

then the Fourier transform $\hat{h}_{r,j}$ of the function

$$h_{r,j}(x) = (1+x^2)g(x, r, j)$$

is given by

$$\hat{h}_{r,j}(t) = \hat{g}(t; r, j) - D^2 \hat{g}(t; r, j).$$

Thus by Fourier inversion and Fubini's Theorem we have

$$(2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\hat{g}(t, s, \omega) - D^2 \hat{g}(t, s, \omega) \right) e^{-i(sr + \omega j + tx)} ds d\omega dt \\ = (1 + x^2) g(x, r, j).$$

Hence we have

$$\begin{aligned} \int_{-\infty}^{\infty} |g(x, r, j)| dx &= \int_A g(x, r, j) dx - \int_{\mathbb{R}-A} g(x, r, j) dx \\ &= \left(\int_A - \int_{\mathbb{R}-A} \right) (1 + x^2) g(x, r, j) dx \\ &= (2\pi)^{-3} \left(\int_A - \int_{\mathbb{R}-A} \right) (1 + x^2)^{-1} \left(\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\hat{g}(t, s, \omega) \right. \\ &\quad \left. - D^2 \hat{g}(t, s, \omega)) e^{-i(sr + \omega j + tx)} ds d\omega dt \right) dx \\ &\leq (2\pi)^{-3} \left(\int_{-\infty}^{\infty} (1 + x^2)^{-1} dx \right) \int_{-\infty}^{\infty} \int_G (|\hat{g}(t, s, \omega)| + |D^2 \hat{g}(t, s, \omega)|) ds d\omega dt \\ &\leq \max_{k=0,2} \int_{-\infty}^{\infty} \int_G |D^k \hat{g}(t, s, \omega)| ds d\omega dt. \end{aligned}$$

This completes the proof of the Lemma. \square

The next lemma is Lemma 2 of Babu and Singh (1984) and is stated here for ready reference.

Lemma 3. *Suppose X_1 satisfies Cramér's condition (3.1). For any $K > 0$, there exists a $0 < \gamma = \gamma(K) < 1$, and $\eta > 0$ such that for almost all sample sequences $\{X_j^i\}$,*

$$\limsup_{n \rightarrow \infty} \sup_{K < |t| < e^m} \left| \frac{1}{n} \sum_{j=1}^n e^{iu(X_j - \bar{X}_n)} \right| < \gamma.$$

The next three lemmas are similar to Lemmas 2 and 3 of Babu and Bai. To state these let c_j satisfy conditions of Proposition 1. Let $d_j = d_{j,n} = c_j n^{-1/2}$. Recall that for some $M' > 1$

$$\sum_1^n d_j = 0, \quad \sum_1^n d_j^2 = 1 \text{ and } \sum_{j=1}^n |d_j|^3 \leq (M'/\sqrt{n}) \quad (3.3)$$

Recall that f_n denotes the characteristic function of $\sum_{j=1}^n (d_j Y_j, Y_j, \delta_j)$. For each fixed r and m , the Fourier transform of

$$F_n(x, r, m) = P\left(\sum_{j=1}^n d_j Y_j \leq x, \sum_{j=1}^n Y_j = r, \sum_{j=1}^n \delta_j = m\right) \quad (3.4)$$

is given by

$$\begin{aligned} h(t; r, m) &= E\left(e^{it\sum_{j=1}^n d_j} I_{(\sum_{j=1}^n Y_j = r, \sum_{j=1}^n \delta_j = m)}\right) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_n(t, s, \omega) e^{-isr - i\omega m} ds d\omega, \end{aligned} \quad (3.5)$$

where f_n is the characteristic function of $\sum_{j=1}^n (d_j Y_j, Y_j, \delta_j)$. Note that

$$f_n(t, s, \omega) = \prod_{j=1}^n f(td_j + s, \omega),$$

where

$$\begin{aligned} f(u, \omega) &= E\left(e^{iuY_1 + i\omega\delta_1}\right) \\ &= e^{-1} \left(1 - e^{i\omega} + e^{i\omega + e^{iu}}\right). \end{aligned}$$

Let for any subset R of $\{1, \dots, n\}$,

$$f_{n,R}(t, s, \omega) = \prod_{j \in R} f(td_j + s, \omega)$$

and for any $0 \leq k \leq 2$, let

$$f_{n,k}(t, s, \omega) = \max\{|f_{n,R}(t, s, \omega)|\},$$

where the maximum is taken over all subsets R of $\{1, \dots, n\}$ of size $n - k$. Finally let

$$d_n(t) = \left| \frac{1}{n} \sum_{j=1}^n e^{itd_j} \right| \quad (3.6)$$

Lemma 4. *For any $0 \leq k \leq 2$,*

$$f_{n,k}(t, s, \omega) \leq \exp(1 - 0.2n(1 - d_n(t))). \quad (3.7)$$

Proof. Observe that

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n |f(td_j + s, \omega)|^2 &= \frac{1}{n} \sum_{j=1}^n E \left(e^{i(td_j + s)(Y_1 - Y_2) + i\omega(\delta_1 - \delta_2)} \right) \\ &= E \left(\left(\frac{1}{n} \sum_{j=1}^n e^{it(Y_1 - Y_2)d_j} \right) e^{is(Y_1 - Y_2) + i\omega(\delta_1 - \delta_2)} \right) \\ &\leq E \left(d_n(t(Y_1 - Y_2)) \right) \\ &\leq P(|Y_1 - Y_2| \neq 1) + d_n(t)P(|Y_1 - Y_2| = 1) \\ &\leq 1 - (1 - d_n(t))P(|Y_1 - Y_2| = 1) \\ &\leq 1 - (1 - d_n(t))0.4. \end{aligned}$$

As $x \leq e^{x-1}$ for all $0 \leq x \leq 1$, for any subset $R \subset \{1, \dots, n\}$ with at least $n - 2$ integers, we have

$$\begin{aligned}
(f_{n,R}(t, s, \omega))^2 &= \prod_{j \in R} |f(td_j + s, \omega)|^2 \\
&\leq \exp \left(2 - n + \sum_{j \in R} |f(td_j + s, \omega)|^2 \right) \\
&\leq \exp \left(2 - n + \sum_{j=1}^n |f(td_j + s, \omega)|^2 \right) \\
&\leq \exp \left(2 - 0.4n(1 - dn(t)) \right).
\end{aligned}$$

This completes the proof. \square

Lemma 5. *For $|t| \leq \sqrt{n}/M'$, we have*

$$d_n(t) = \left| \frac{1}{n} \sum_{j=1}^n e^{itd_j} \right| \leq 1 - \frac{1}{3n} t^2, \quad (3.8)$$

and for $0 \leq k \leq 2$, $|s| \leq \pi$, $|\omega| \leq \pi$, $|t| \leq \sqrt{n}/M'$,

$$f_{n,2}(t, s, \omega) \leq \exp(1 - \frac{1}{15}t^2). \quad (3.9)$$

Proof. For $|t| \leq \sqrt{n}/M'$,

$$\begin{aligned}
d_n(t) &= \left| 1 - \frac{t^2}{2n} + \frac{1}{n} \sum_{j=1}^n \left(e^{itd_j} - 1 - itd_j + \frac{t^2}{2} d_j^2 \right) \right| \\
&\leq 1 - \frac{1}{2n} t^2 + \frac{M'}{6} |t|^3 n^{-3/2} \\
&\leq 1 - \frac{1}{3n} t^2.
\end{aligned}$$

The second part follows from (3.7). \square

Lemma 6. *Suppose $|t| \leq \log n$, $|s| \leq \pi$, $|\omega| \leq \pi$ and $s^2 + \omega^2 \geq (\log n)^2 n^{-1}$. Then there exist constants k_1 and k_2 depending only on M' such that for $0 \leq k \leq 2$,*

$$f_{n,k}(t, s, \omega) \leq k_1 \exp(-k_2(\log n)^2). \quad (3.10)$$

Proof. As $E(Y_1^3) = 5$,

$$\left| e^{-iu-ivp} E\left(e^{iuY_1+iv\delta_1}\right) - 1 + \frac{1}{2}(u,v)\Sigma_0(u,v)' \right| \leq (u^2 + v^2)^{3/2}.$$

Since

$$(u,v)\Sigma_0(u,v)' \geq q(\frac{1}{2} - q)(u^2 + v^2),$$

there exists a positive $\eta < \frac{q}{16}(\frac{1}{2} - q)$ such that for all $|u| < 4\eta$, $|v| < 4\eta$, we have

$$\begin{aligned} \frac{1}{2} &\leq \left| E\left(e^{iuY_1+iv\delta_1}\right) \right| \leq 1 - 2^{-1}(u,v)\Sigma_0(u,v)' + (u^2 + v^2)^{3/2} \\ &\leq 1 - \frac{q}{2}(\frac{1}{2} - q)(u^2 + v^2) + (u^2 + v^2)^{3/2} \\ &\leq 1 - \left(\frac{q}{2}(\frac{1}{2} - q) - 4\eta\right)(u^2 + v^2) \\ &\leq 1 - \eta(u^2 + v^2) \\ &\leq e^{-\eta(u^2 + v^2)}. \end{aligned} \tag{3.11}$$

If $|t| \leq (\log n)$ then $|td_j| \leq (M')^{1/3}n^{-1/6}\log n \leq \eta$ for all $n > n(M', \eta)$. In this case, if $|s| < 2\eta$, $|\omega| < 2\eta$, then $|s + td_j| \leq 3\eta$ for all $n \geq n(M', \eta)$. So by (3.11),

$$\begin{aligned} f_{n,k}(t, s, \omega) &\leq 4|f_n(t, s, \omega)| \\ &\leq 4e^{-\eta n(s^2 + \omega^2) - \eta t^2} \\ &\leq 4e^{-\eta n(s^2 + \omega^2)} \\ &\leq 4e^{-\eta(\log n)^2} \end{aligned}$$

provided $s^2 + \omega^2 \geq (\log n)^2 n^{-1}$ and $|s| < 2\eta$, $|\omega| < 2\eta$.

To establish inequality (3.10) for the remaining s and ω , note that b given by

$$b(u) = E(e^{iuY_1} | Y_1 > 0)$$

is the characteristic function of a lattice distribution of span 1. So for $0 < \eta < \pi/10$,

$$\begin{aligned} \left| E(e^{iuY_1+iv\delta_1}) \right| &= \left| q + pe^{iv}r(u) \right| \\ &\leq q + p|r(u)| < 1 - \gamma \end{aligned}$$

for some $0 < \gamma < 1$, whenever $\eta < |u| < \pi + \eta$. Hence for $|t| \leq \log n$, $2\eta \leq |s| < \pi$, all ω and for all large n ,

$$|f(td_j + s, \omega)| \leq 1 - \gamma.$$

Now consider the case $|s| < 2\eta$ and $|v| \geq 2\eta$. Let $0 < \theta < \eta$ be such that $|1 - r(u)| < \eta^2 q$ whenever $|u| < \theta$. Hence as $0 < \eta < 1$,

$$\begin{aligned} \left| E\left(e^{iuY_1+iv\delta_1}\right) \right| &= |q + pe^{iv}r(u)| \\ &\leq |q + pe^{iv}| + p|1 - r(u)| \\ &\leq |q + pe^{iv}| + \eta^2 pq \\ &= (1 - 2pq(1 - \cos u))^{1/2} + \eta^2 pq \\ &= (1 - 2pq(1 - \cos \eta))^{1/2} + \eta^2 pq \\ &= 1 + pq(\cos 2\eta - 1 + \eta^2) \\ &\leq 1 - \eta^2 pq \leq 1 - \theta^2 pq. \end{aligned}$$

Thus if $|t| \leq \log n$ and $|s| < \theta/2$ then $|rd_j + s| < \theta$ for all large n . So if $|\omega| > 2\eta$, $0 \leq k \leq 2$, then

$$f_{n,k}(t, s, \omega) \leq k_3 \rho^n$$

for some $k_3 > 0$ and $0 < \rho < 1$. This completes the proof of the lemma. \square

4. SECOND ORDER CORRECTION

Let $\{X_1, \dots, X_n\}$ be i.i.d. random variables with mean μ and variance σ^2 . Suppose X_1 satisfies Cramér's condition

$$\limsup_{|t| \rightarrow \infty} |E(e^{itX_1})| < 1. \quad (4.1)$$

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$, $s_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $C_j = C_{j,n} = (X_j - \bar{X}_n)/s_n$. Let Y_1, Y_2, \dots be a sequence of i.i.d. Poisson random variables with mean 1. Put $\delta_i = 1$ if $Y_i > 0$ and =0 otherwise. Let $N = \sum_{i=1}^n Y_i$, $T_n = \sum_{i=1}^n \delta_i$, $q = e^{-1} = P(Y = 0)$ and $p = 1 - e^{-1}$. We shall obtain Edgeworth expansions for $\frac{1}{\sqrt{N}} \sum_{j=1}^n C_j Y_j$ given $T_n = m$ and X_1, \dots, X_n . Then use this result to establish second order correctness for the Poisson Bootstrap in the simple case of $\sqrt{n}(\bar{X} - \mu)/\sigma$. This result can be extended to statistics which can be represented as a smooth function of a multivariate mean. Second order correctness of Poisson Bootstrap for such models are discussed in Section 5.

We now state the main theorem.

Theorem 4.1. *Let $E|X_1|^5 < \infty$ and the characteristic function of X_1 satisfies Cramér's condition (4.1). If $m - np$ is bounded, then*

$$\begin{aligned} P\left(\frac{1}{\sqrt{N}} \sum_{i=1}^n (X_i - \bar{X}_n) Y_i \leq xs_n \mid T_n = m; X_1, \dots, X_n\right) \\ - P\left(\frac{1}{\sqrt{N}} \sum_{i=1}^n (X_i - E(X_1)) \leq x\sigma\right) = o_p(n^{-1}), \end{aligned} \quad (4.2)$$

uniformly in x .

Remark 1. With truncation and additional analysis, the moment condition on X_1 can be relaxed.

To prove Theorem 4.1 we first establish some notation and a preliminary proposition.

Let ϕ denote the standard normal density, φ_0 denote the bivariate normal density with zero mean vector and dispersion matrix

$$\Sigma_0 = \begin{pmatrix} 1 & q \\ q & pq \end{pmatrix},$$

where $q = e^{-1}$ and $p = 1 - q$. Define

$$\begin{aligned} P_n(x) &= \left(\frac{1}{6} \sum_{j=1}^n C_j^3 \right) (x^3 - x) \\ Q_n(x, y, \omega) &= P_n(x) + \frac{1}{6} E \left(Z' \Sigma_0^{-1}(y, \omega)' \right)^3 \\ &= \frac{1}{2} \left(E((Z' \Sigma_0^{-1}(y, \omega)') (Y - 1)^2 pq + (q - p)\delta) / (q(p - q)) \right) \\ &\quad + \frac{1}{2} (x^2 - 1) E \left((Y - 1)^2 Z' \Sigma_0^{-1}(y, \omega)' \right), \end{aligned}$$

where Y is a Poisson random variable with mean 1, $\delta = I_{\{Y>0\}}$ and $Z = (Y - 1, \delta - p)'$.

For brevity fix $X_1 \dots X_n$ and define

$$y_r = (r - n)n^{-1/2}, \quad \omega_m = (m - np)n^{-1/2},$$

and

$$F_n(x, r, m) = P \left(\sum_{i=1}^n C_i Y_i \leq x\sqrt{n}, \quad N = r, \quad T_n = m \right).$$

Note that $\sum_{i=1}^n C_i = 0$ and $\sum_{i=1}^n C_i^2 = n$.

Proposition 1. Suppose for any $K > 0$, there exists a $0 < \gamma < 1$ such that

$$\limsup_{n \rightarrow \infty} \sup_{K \leq |t| \leq n^3} \left| \frac{1}{n} \sum_{j=1}^n e^{itC_j} \right| < \gamma.$$

Suppose for some $M > 1$, $\sum_{i=1}^n |C_i|^5 < nM$. Then uniformly in x and r , we have

$$\begin{aligned} nP\left(\sum_{i=1}^n C_i Y_i \leq x\sqrt{n}, N=r, T_n=m\right) \\ = \int_{-\infty}^x \Psi_n(v, y_r, \omega_m) dv + O(n^{-3/2}), \end{aligned} \quad (4.3)$$

where

$$\Psi_n(v, y, \omega) = \phi(v)\varphi_0(y, \omega) \left(1 + \frac{1}{\sqrt{n}}Q_n(v, y, \omega) + \frac{1}{n}Q_{n_1}(v, y, \omega)\right) \quad (4.4)$$

Q_m is a fourth degree polynomial in v, y, ω whose coefficients are bounded by a constant depending only on M .

We use the lemmas proved in Section 3 in proving the proposition.

Proof of Proposition 1. Since for any $\epsilon > 0$,

$$\int_{x-\epsilon}^{x+\epsilon} \phi(u) du = 0(\epsilon)$$

uniformly in x , we have by Lemmas 1 and 2, with $\epsilon = n^{-2}$,

$$\begin{aligned} \left| nF_n(x, r, m) - \int_{-\infty}^x \Psi_n(v, y_r, \omega_m) dv \right| &\ll n^{-2} \\ + \max_{0 \leq j \leq 2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-cn^{5/2}}^{-cn^{5/2}} &\left| D^j \left(n f_n(t, s, \omega) e^{-isr-i\omega m} - \hat{\Psi}_n(t, s, \omega) \right) \right| dt ds d\omega, \end{aligned} \quad (4.5)$$

where $\hat{\Psi}_n$ and f_n denote the characteristic functions of Ψ_n and $\sum_{i=1}^n \left(\frac{1}{\sqrt{n}} C_j Y_j, Y_j, \delta_j \right)$. Note that for each r and m , the Fourier transform of $F_n(\cdot, r, m)$ is given by

$$\hat{F}_n(t, r, m) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_n(t, s, \omega) e^{-isr-i\omega m} ds d\omega. \quad (4.6)$$

To estimate the last integral in (4.5), we try to divide the range of integration into 4 possibly overlapping regions:

- (i) $|t| \leq \log n, |s| \leq (\log n)n^{-1/2}, |\omega| \leq (\log n)n^{-1/2}$
- (ii) $|t| \leq \log n, \omega^2 + s^2 \geq (\log n)^2 n^{-1}, |\omega| \leq \pi, |s| \leq \pi$

- (iii) $\log n < |t| \leq \sqrt{n}/M$, $|s| \leq \pi$ and $|\omega| \leq \pi$
- (iv) $\left(\sqrt{n}/M\right) \leq |t| \leq cn^{5/2}$, $|s| \leq \pi$ and $|\omega| \leq \pi$,

where c is the constant in (3.1).

We expand $nD^j f_n(r, s, \omega)$ in region (i) and estimate the integral in (4.5) as in the proof of Theorem 9.9 of Bhattacharya and Ranga Rao (1986). Lemma 6 is used for region (ii), Lemma 5 is used for region (iii) and Lemmas 3 and 4 are used for region (iv) to estimate the integral. These estimates lead to a bound of $O(n^{-3/2})$ for (4.5), completing the proof.

Proof of Theorem 4.1. Using Proposition 1, we shall first show that uniformly in x ,

$$\begin{aligned} P\left(\frac{1}{\sqrt{N}} \sum_{i=1}^n (X_i - \bar{X}) Y_i \leq xs_n \middle| T_n = m, X_1, \dots, X_n\right) \\ = \Phi(x) + \frac{1}{6\sqrt{n}} \gamma_3 (1 - x^2) \phi(x) + O_p\left(\frac{1}{n}\right), \end{aligned} \quad (4.7)$$

where Φ is the standard normal distribution function and

$$\gamma_3 = \sigma^{-3} E(X_1 - \mu)^3.$$

Let

$$H_n = \left\{ \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i - 1) \right| \leq \log n \right\}.$$

Then by a moderate deviation result

$$1 - P(H_n) = O(n^{-10}). \quad (4.8)$$

Suppose $\sum_{j=1}^n |X_j - \bar{X}_n|^5 \leq nMs_n$ for some $M > 1$, then by Proposition 1,

$$\sqrt{n}P\left(\frac{1}{\sqrt{N}} \sum_{j=1}^n (X_j - \bar{X}_n) Y_j \leq xs_n, T_n = m\right)$$

$$\begin{aligned}
&= \sum_{|r-n| \leq \sqrt{n} \log n} \sqrt{n} P \left(\frac{1}{\sqrt{r}} \sum_{j=1}^n C_j Y_j \leq x, N = r, T_n = m \right) + O(n^{-9}) \\
&= \frac{1}{\sqrt{n}} \sum_{|r-n| \leq \sqrt{n} \log n} \int_{-\infty}^{x\sqrt{r/n}} \Psi_n(v, y_r, \omega_m) dv + O(n^{-1}) \\
&= \frac{1}{\sqrt{n}} \sum_{|r-n| \leq \sqrt{n} \log n} \int_{-\infty}^x \Psi_n \left(u \left(1 + \frac{1}{\sqrt{n}} y_r \right)^{1/2}, y_r, \omega_m \right) \left(1 + \frac{1}{\sqrt{n}} y_r \right)^{1/2} du + O(n^{-1}) \\
&= \int_{-\infty}^x \left(\int_{|y| < \log n} \Psi_n \left(u \left(1 + \frac{1}{\sqrt{n}} y \right)^{1/2}, y, \omega_m \right) \left(1 + \frac{1}{\sqrt{n}} y \right)^{1/2} dy \right) du + O(n^{-1}) \quad (4.9)
\end{aligned}$$

By Theorem 13 on local Edgeworth expansion on pages 205-206 of Petrov (1975), we have

$$\begin{aligned}
\sqrt{npq} P(T_n = m) &= \phi(\omega_m) + \sum_{j=1}^2 n^{-j/2} q_j(\omega_m) \phi(\omega_m) + O(n^{-1}) \\
&= \phi(\omega_m) + n^{-1/2} q_1(\omega_m) \phi(\omega_m) + O(n^{-1}) \quad (4.10)
\end{aligned}$$

where q_1 and q_2 are polynomials and

$$q_1(y) = \frac{1}{6} E(\delta_1 - p)^3 (pq)^{-3/2} (y^3 - 3y).$$

The estimate (4.7) follows from (4.8), (4.9) and (4.10) after some simple algebra. The Theorem now follows from the standard Edgeworth expansion

$$P \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - M) \leq \sigma x \right) = \Phi(x) + \frac{1}{\sqrt{n}} \gamma_3 (1 - x^2) \phi(x) + O(n^{-1}),$$

and

$$P \left(\sum_{j=1}^n |X_j - \bar{X}_n|^5 > nMs_n \right) = O \left(\frac{1}{n} \right) \text{ for some } M > 1.$$

This completes the proof.

5. SMOOTH FUNCTIONAL MODEL

Proposition 1 and Theorem 3.1 can be extended to multivariate case and to statistics which can be expressed as smooth functions of multivariate means. Let X_1, \dots, X_n be a sequence of i.i.d. random vectors with mean μ and dispersion Σ . Let Σ_n denote the sample dispersion matrix of X_1, \dots, X_n . With some additional effort and using the ideas and lemmas from Bai and Rao (1991) and Babu and Bai (1996), we can establish the following results.

Theorem 5.1. *Suppose the characteristic function of X_1 satisfies Cramér's condition and $E\|X_1\|^4$. Let H be a 3-times continuously differentiable function in a neighborhood of μ . Let $\ell(y)$ denote the vector of first order partial derivatives at y , and $\ell(\mu) \neq 0$. If $m - n(1 - e^{-1})$ is bounded, then for almost all sample sequences X_1, X_2, \dots , we have*

$$\begin{aligned} & \sup_z \sqrt{n} \left| P \left(\frac{\sqrt{N}(H(\frac{1}{N} \sum_{i=1}^n X_i Y_i) - H(\bar{X}_n))}{\sqrt{\ell'(\bar{X}_n) V_n^2 \ell(\bar{X}_n)}} \leq z \middle| T_n = m; X_1, \dots, X_n \right) \right. \\ & \quad \left. - P \left(\sqrt{n} \left(H(\bar{X}_n) - H(\mu) \right) \leq z \sqrt{\ell'(\mu) \Sigma \ell(\mu)} \right) \right| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

The next two results are more suitable for applications to studentized statistics,

Theorem 5.2. *Let $\{X_n\}$ be as in Theorem 5.1. Suppose the function H is three times continuously differentiable in a neighborhood of the origin and $H(0) = 0$. If $m - n(1 - e^{-1})$ is bounded, then for almost all sample sequences X_1, X_2, \dots , we have*

$$\begin{aligned} & \sup_z \sqrt{n} \left| P \left(\frac{\sqrt{N}(H(\frac{1}{N} \sum_{i=1}^n (X_i - \bar{X}_n) Y_i) - \mu)}{\sqrt{\ell'(0) \Sigma_n^2 \ell(0)}} \leq z \middle| T_n = m; X_1, \dots, X_n \right) \right. \\ & \quad \left. - P \left(\sqrt{n} H(\bar{X}_n) - \mu \leq z \sqrt{\ell'(0) \Sigma \ell(0)} \right) \right| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

It is easily seen that the second order correctness of the sequential bootstrap pivot such as

$$\pi_N^* = \sqrt{N} \left(\sum_{i=1}^n (X_i - \bar{X}_n) Y_i \right) / s_n$$

given that $T_n := \left(\sum_{j=1}^n I_{\{Y_j > 0\}} \right) = m$ follows from Theorem 5.1. The one-term correction captures the skewness of the underlying distribution.

As the most commonly used statistics, especially the studentized versions, are of the type

$$t_n = \sqrt{n} \left(H(\bar{X}) - H(\eta) \right) / \xi \left(\frac{1}{n} \sum_{i=1}^n \lambda(X_i) \right), \quad (5.1)$$

where λ is a function on $\mathbb{R}^k \rightarrow \mathbb{R}^r$ and ξ is a smooth real-valued function on \mathbb{R}^r . The classical Student's t is an example of this type of statistic. If X_i are univariate, then

$$t_n = \frac{\sqrt{n}(\bar{X}_n - \eta)}{s_n},$$

satisfies (5.1) with $H(x) = x$, $\lambda(x) = (x^2, x)$, $\nu(x, y) = \max(0, (x - y^2))^{1/2}$ and $s_n^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{X})^2$. The version corresponding to (5.1) under the Poisson scheme is generally of the type,

$$t_n(Y) = \sqrt{N} \left(H \left(\frac{1}{N} \sum_{i=1}^n X_i Y_i \right) - H(\bar{X}_n) \right) / \nu \left(\frac{1}{N} \sum_{i=1}^n \lambda(X_i) Y_i \right).$$

Let \sum_n denote the sample dispersion of X_1, \dots, X_n . As in Theorem 4 of Babu and Singh (1984), we can derive:

Theorem 5.3. *Let*

$$\begin{aligned} \nu(E(\lambda(X_1))) &= \sqrt{\ell'(\eta)\Sigma\ell(\eta)}, \\ \nu(\bar{X}) &= \sqrt{\ell'(\bar{X})\Sigma_n\ell(\bar{X})}, \end{aligned}$$

and let $L(X_i)$ be a linearly independent sub collection of $(X_i, \lambda(X_i))$ with the property that all the coordinates of $(X_i, \lambda(X_i))$ can be expressed as linear combinations of those of $L(X_i)$. If the characteristic function of $L(X_1)$ satisfies Cramèr's condition $E\|L(X_1)\|^4 < \infty$, and if $m - np$ is bounded then

$$\sup_a \sqrt{n} |P(t_n(Y) \leq a | T_n = m, X_1, \dots, X_n) - P(t_n \leq a)| \rightarrow 0,$$

as $n \rightarrow \infty$, for almost all sample sequences $\{X_j\}$.

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